PURE NASH EQUILIBRIA IN ONLINE FAIR DIVISION

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In online fair division, items arrive one by one and are allocated to agents via two simple mechanisms: LIKE and BALANCED LIKE. We study pure Nash equilibria of these two mechanisms.

Model and mechanisms

Allocation Instance I: agents a_1 to a_n, indivisible items o_1 to o_m, utility $u_{ij} \in \mathbb{Q}^{\geq 0}$ for each a_i and o_j, ordering $o = (o_1, \ldots, o_m)$ and mechanism M

Setting: At moment j, item o_j arrives according to o, each a_i reports (or *bids*) a value $v_{ij} \in \mathbb{Q}^{\geq 0}$ for o_j and M allocates o_j to an agent that is considered *feasible*.

Let π_j be an allocation of o_1 to o_{j-1}. Given π_j, the probability $p_i(o_j)$ of agent a_i for item o_j is $\frac{1}{n_j}$ where n_j is the number of feasible agents.

The LIKE mechanism: given π_j, agent a_i is feasible for o_j if $v_{ij} > 0$.
The BALANCED LIKE mechanism: given π_j, agent a_i is feasible for o_j if $v_{ij} > 0$ and has received fewest items in π_j among those agents with positive bids for o_j.

Strategy-proofness

Online Strategy-Proofness: For each round j and π_j, no agent a_i can increase their outcome of $u_i(\pi_j) + p_i(o_j) \cdot u_{ij}$ supposing π_j is *fixed* and no information about future items is known.

Strategy-Proofness: For each round j and π_j, no agent a_i can increase their outcome of $u_i(\pi_j) + \sum_{k=j}^{m} p_i(o_k) \cdot u_{ik}$ supposing π_j is *fixed* and all information about future items is known.

Online Group Strategy-Proofness: For each round j and π_j, no group G can increase their outcome of $\sum_{a_i \in G} u_i(\pi_j) + p_i(o_j) \cdot u_{ij}$ supposing past bids are *fixed* and no information about future items is known.

Group Strategy-Proofness: For each round j and π_j, no group G can increase their outcome of $\sum_{a_i \in G} u_i(\pi_j) + \sum_{a_i \in G} p_i(o_j) \cdot u_{ij}$ supposing past bids are *fixed* and all information about future items is known.

Example (Online vs offline strategic behavior): Agents a_1, a_2, items o_1, o_2, utilities $u_{11} = 1, u_{12} = 2, u_{21} = 2, u_{22} = 1$, $o = (o_1, o_2)$ and BALANCED LIKE.

o	o_1	o_2
a_1	1	2
a_2	2	1

1. With knowledge of o_1 and o_2, a_1 increases their outcome from $\frac{3}{2}$ to 2 if they bid strategically 0 for o_1.
2. With knowledge of o_1 only, each agent bids their sincere utility for o_1.

mechanism	SP	OSP	GSP	OGSP
		general utilities		
LIKE	✓	✓	×	×
BALANCED LIKE	×	✓	×	×
		binary utilities		
LIKE	✓	✓	✓	✓
BALANCED LIKE	×	✓	×	✓

Table 1: Axiomatic results.

Pure Nash equilibria

Group PNE: For each j, π_j, no group G of agents has an incentive to misreport their bids for o_j to o_m and increase $\sum_{a_i \in G} u_i(\pi_j) + \sum_{a_i \in G} \sum_{k=j}^{m} p_i(o_k) \cdot v_{ik}$ supposing all bids of agents of all other groups are fixed.

Online Group PNE: For each j, π_j, no group G of agents has an incentive to misreport their bids for o_j and increase $\sum_{a_i \in G} u_i(\pi_j) + \sum_{a_i \in G} p_i(o_j) \cdot v_{ij}$ supposing all bids of agents of all other groups are fixed.

Note: Competitive PNE and Online PNE suppose each agent is in a group alone.

Computing equilibria

(ONLINE) GROUP PURE NASH EQUILIBRIUM
Input: instance I, mechanism $\mathcal{M}$ and groups $G_1, \ldots, G_k$
Output: a (online) group pure Nash equilibrium of $\mathcal{A}$ with $\mathcal{M}$

Theorem 1: With the BALANCED LIKE mechanism and 0/1 utilities, computing (online) PNE is in NP-hard.

Unique weak PNE [3].

o	o_1	o_2
a_1	1	2
a_2	2	1

Unique strict PNE [3].

o	o_1	o_2	o_3
a_1	1	1	1
a_2	0	1	0
a_3	1	0	1

Proof main steps:

1. Deciding if an agent receives an item with > 0 probability is NP-hard [1].
2. Computing a weak (strict) PNE is at least as hard as deciding if an agent gets an item with 0 (> 0) expected probability.

Theorem 2: With the LIKE mechanism and general utilities, computing (online) GPNE is in P.

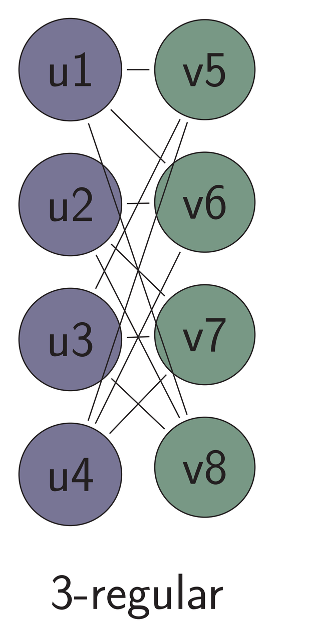
Proof main steps: We present a simple iterative algorithm in which agents commit to their optimal strategies at earliest iterations.

1. For each item o_j,
   For each group G_l with an agent whose strategy for o_j is not computed,
   compute the set $S \subseteq G_l$ of agents that maximize $\frac{\sum_{a_i \in S} u_{ij}}{|S| + r_{jl}}$.
2. The computed profile for o_j and $G_1, \ldots, G_k$ is a (online) GPNE.
3. The collection of computed (online) GPNE for items o_1 to o_m is a (online) GPNE because LIKE is Markovian.

Counting equilibria [2]

#(ONLINE) GROUP PURE NASH EQUILIBRIUM
Input: instance I, mechanism $\mathcal{M}$ and groups $G_1, \ldots, G_k$
Output: number of (online) GPNE of $\mathcal{A}$ with $\mathcal{M}$

Theorem 3: With the BALANCED LIKE mechanism and 0/1 utilities, counting (online) GPNE is in #P-hard.



3-regular bipartite graph G

o	v_1	v_2	v_3	v_4	a_1	b_1	a_2	b_2	a_3	b_3	a_4	b_4
u_1^1	1	0	0	0	1	1	0	0	0	0	0	0
u_1^2	0	1	0	0	1	1	0	0	0	0	0	0
u_1^3	0	0	0	1	1	1	0	0	0	0	0	0
u_2^1	0	1	0	0	0	0	1	1	0	0	0	0
u_2^2	0	0	1	0	0	0	1	1	0	0	0	0
u_2^3	0	0	0	1	0	0	1	1	0	0	0	0
u_3^1	1	0	0	0	0	0	0	0	1	1	0	0
u_3^2	0	0	1	0	0	0	0	0	1	1	0	0
u_3^3	0	0	0	1	0	0	0	0	1	1	0	0
u_4^1	1	0	0	0	0	0	0	0	0	0	1	1
u_4^2	0	1	0	0	0	0	0	0	0	0	1	1
u_4^3	0	0	1	0	0	0	0	0	0	0	1	1

instance I_G

Theorem 4: With the LIKE mechanism and general utilities, counting (online) GPNE is in #P-hard.

FOR FURTHER INFORMATION

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